University of Minnesota: Twin Cities

CE 8351: Analytical Modeling in Civil Engineering

Project 3: Hodograph Method

Phreatic Surface Over Symmetric Lateral Trenches

Chris [redacted] Logston

April 12$^{\text{th}}$, 2015

Contents

0) Introduction

0.1) Scenario

As shown in figure 1, the case is presented in which a permeable region is bounded by two lateral drain systems extending to $\pm\infty$. Due to infiltration across a certain region, a phreatic surface forms above the drains within the porous medium. Given that both the discharge potential and streamline along this surface are variable, a free boundary exists. Therefore, previous methods of developing a flow-net along a vertical plane of analysis (as shown) do not apply. However, given that the wells each act as points of inversion (where Ψ approaches $\pm\infty$), the hodograph method can be used as a substitute flow-net development technique.

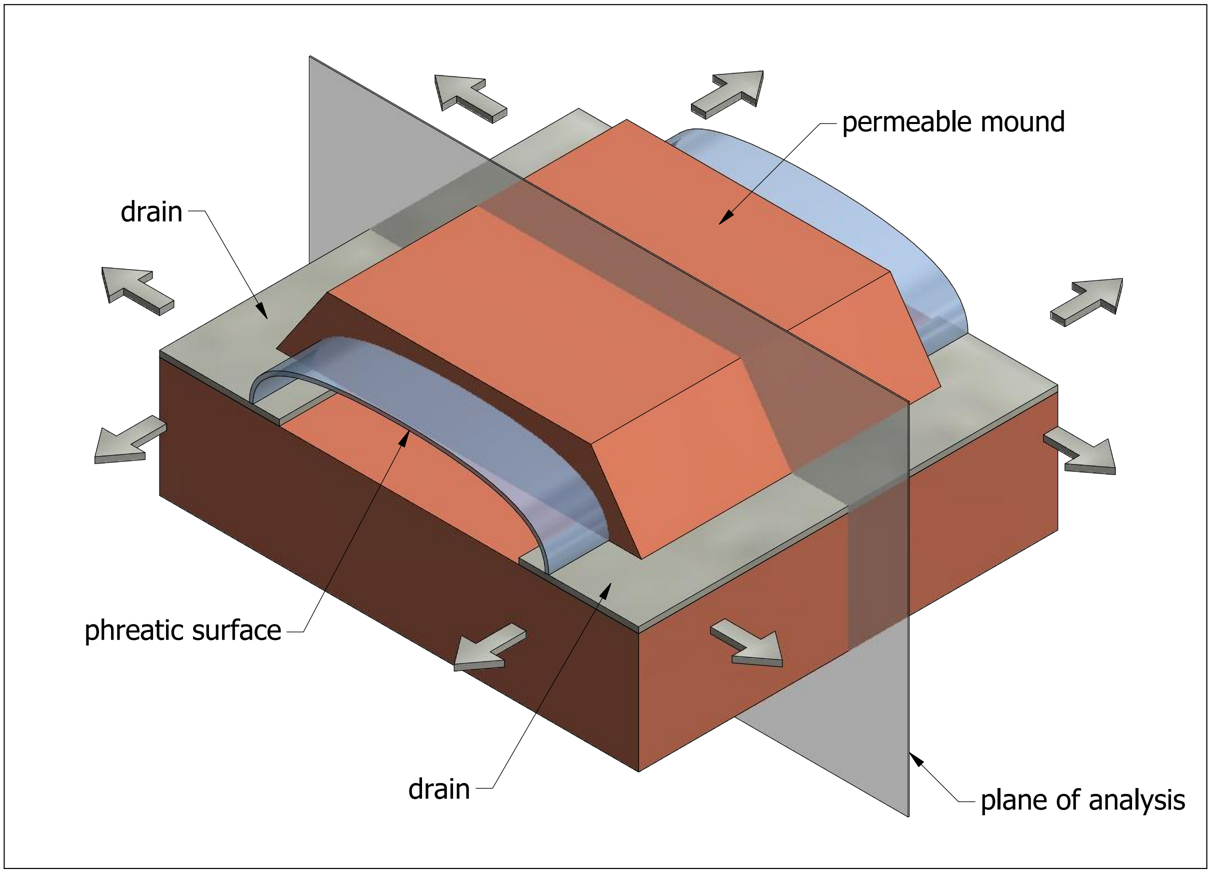

Figure 1: Scenario under inspection

0.2) Setup

As shown in figure 2, the relevant properties within the physical analysis plane are as follows:

- **N**: infiltration rate;
- **2b**: region over which infiltration occurs;
- **L**: distance between drains edges;
- as well as **k**: the hydraulic conductivity of the porous medium.

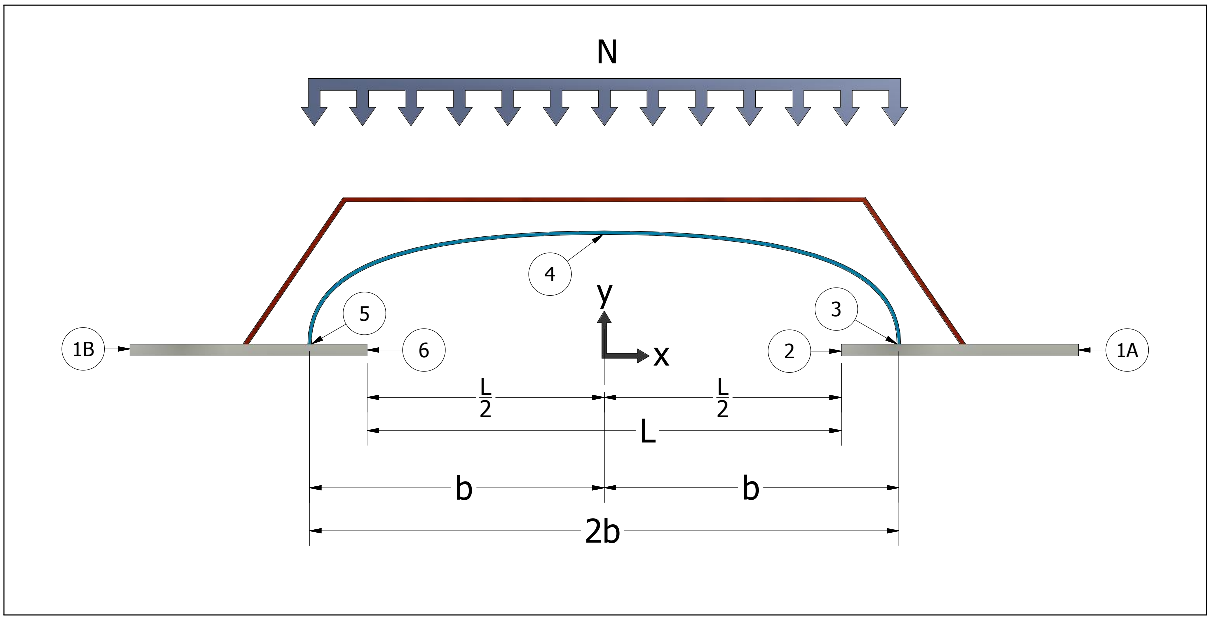

Figure 2: Setup of physical flownet (z) plane

Also, several points are labeled that are of importance to the conformal mapping process carried out in the Hodograph method. They are:

- points **1B** and **1A**, both drain extents at ∞;
- points **5** and **3**, the horizontal extents of the phreatic surface, aligned with $x = -b$ and $x = b$ respectively;
- points **6** and **2**, where the drains edges terminate.

0.3) Given Solution

As presented during the University of Minnesota, Twin Cities course *CEGE 8351: Analytical Modeling in Civil Engineering* during the Spring, 2016 term by Professor Otto Strack, PhD, the complex potential for this symmetric case in which both drains are at the same elevation is as follows:

$$\Omega = -i|A|\frac{N}{N-k}\zeta + i|B|\frac{k}{N-k}\sqrt{(\zeta-1)(\zeta+1)} \tag{1}$$

Where:

- both $|A|$ and $|B|$ are functions of hydraulic conductivity and infiltration:

$$|A| = \frac{L}{2}k\frac{\sqrt{k-N}}{\sqrt{k+N}} \tag{2}$$

$$|B| = \frac{L}{2}N\frac{\sqrt{k-N}}{\sqrt{k+N}} \tag{3}$$

- ζ is complex location in the ζ-plane where:
  - horizontal (real) and vertical (imaginary) components are distinguished as:

$$\zeta = \xi + i\eta \tag{4}$$

  - all locations can be re-mapped into the physical z-plane as follows:

$$z = \frac{-|A|\zeta}{k-N} + \frac{|B|}{k-N}\sqrt{(\zeta-1)(\zeta+1)} \tag{5}$$

This solution is used to develop the flownets in both the ζ and z planes in section 3.

1) Verification of Solution

As discussed in the development of the solution above, the following boundary conditions are set:

- streamline at point 3: $\Psi(\zeta_3) = (-N)(b)$;
- streamline at point 5: $\Psi(\zeta_5) = (-N)(-b)$;
- streamline at point 4: $\Psi(\zeta_4) = 0$.
- Φ at all $\zeta = \xi + 0i$ for $\xi < -1, \xi > 1$: $\Re[\Omega] = 0$

Given that 3 and 5 mark the points of inversion in the physical z-plane, their ζ values are 1 and -1, respectively. Thus their associated boundary conditions are checked using equation 1.

- Starting with point 3, where $\zeta_3 = -1 + 0i$
  - the stream line can expressed as:

$$\Psi(\zeta_3) = \Im[\Omega(\zeta_3)] = \Im[\Omega(-1)] = \Im\left[-i|A|\frac{N}{N-k}(-1) + i|B|\frac{k}{N-k}\sqrt{(-1-1)(-1+1)}\right]$$

$$= \Im\left[i|A|\frac{N}{N-k}\right] = |A|\frac{N}{N-k} = \boxed{\frac{|A|N}{N-k}}$$

  - and the boundary condition can be re-expressed using $z = b + 0i$ as follows:

$$(-N)(b) = (-N)(z_3) = (-N)\left[\frac{-|A|(-1)}{k-N} + \frac{|B|}{k-N}\sqrt{(-1-1)(-1+1)}\right] = (-N)\left[\frac{|A|}{k-N}\right]$$

$$= N\left[\frac{|A|}{N-k}\right] = \boxed{\frac{|A|N}{N-k}}$$

- Continuing with point 5, where $\zeta_5 = 1 + 0i$
  - the streamline can be expressed as:

$$\Psi(\zeta_5) = \Im[\Omega(\zeta_5)] = \Im[\Omega(1)] = \Im\left[-i|A|\frac{N}{N-k}(1) + i|B|\frac{k}{N-k}\sqrt{(1-1)(1+1)}\right]$$

$$= \Im\left[-i|A|\frac{N}{N-k}\right] = -|A|\frac{N}{N-k} = \boxed{\frac{-|A|N}{N-k}}$$

  - and the boundary condition can be re-expressed using $z = -b + 0i$ as follows:

$$(-N)(-b) = (-N)(z_5) = (-N)\left[\frac{-|A|(1)}{k-N} + \frac{|B|}{k-N}\sqrt{(1-1)(1+1)}\right] = (-N)\left[\frac{-|A|}{k-N}\right]$$

$$= (-N)\left[\frac{|A|}{N-k}\right] = \boxed{\frac{-|A|N}{N-k}}$$

- Also, checking at point 4, where $\zeta_4 = 0 + 0i$
  - the streamline is checked:

$$\Psi(\zeta_4) = \Im[\Omega(\zeta_4)] = \Im\left[-i|A|\frac{N}{N-k}(0) + i|B|\frac{k}{N-k}\sqrt{(0-1)(0+1)}\right]$$

$$= \Im\left[i|B|\frac{k}{N-k}\sqrt{(-1)(1)}\right] = \Im\left[i|B|\frac{k}{N-k}\sqrt{(-1)}\right]$$

$$= \Im\left[i|B|\frac{k}{N-k}i\right] = \Im\left[-1|B|\frac{k}{N-k}\right] = \boxed{0}$$

  - It should be noted that this verification assumes $k > N$, wherein both $|A|$ and $|B|$ are only real.

- Finally, the complex potential along the following boundary is checked to be entirely imaginary, such that $\Phi = 0$ throughout:

$$\Re[\Omega(\xi + 0i)] = 0 \text{ where } \xi < -1, \xi > 1$$

  - regarding eq. 1, replacing ζ with ξ, its only non-zero component:

$$\Omega = -i|A|\frac{N}{N-k}\xi + i|B|\frac{k}{N-k}\sqrt{(\xi-1)(\xi+1)}$$

  - it can be seen that for all $\xi < -1$ and $\xi > 1$, all terms are imaginary:

$$\Re\left[-i|A|\frac{N}{N-k}\xi\right] = 0$$

$$\Re\left[i|B|\frac{k}{N-k}\sqrt{(\text{something} > 0)}\right] = 0$$

  - both Φ and Ψ are plotted for an array of ζ values along this boundary range in figure 3.

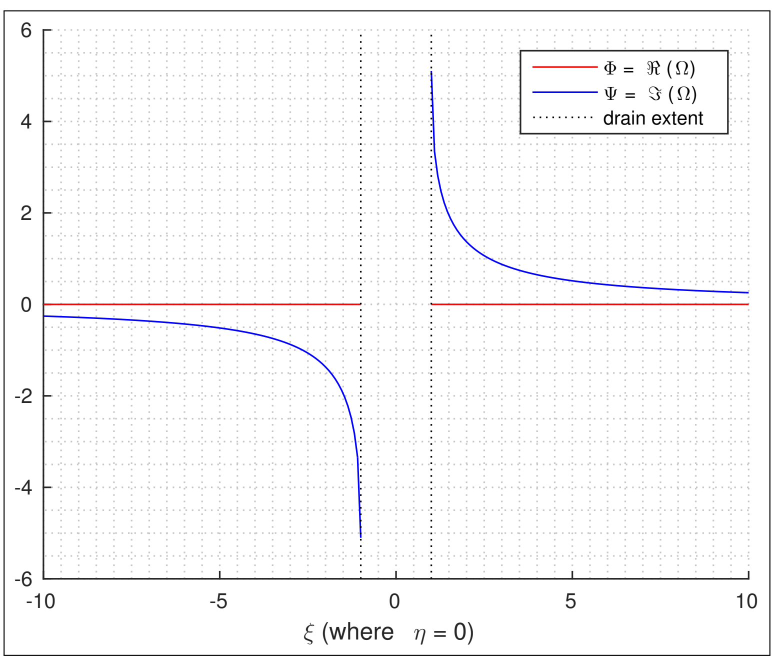

Figure 3: Boundary check in ζ-plane

– Figure 4 presents the same check in the z-plane.

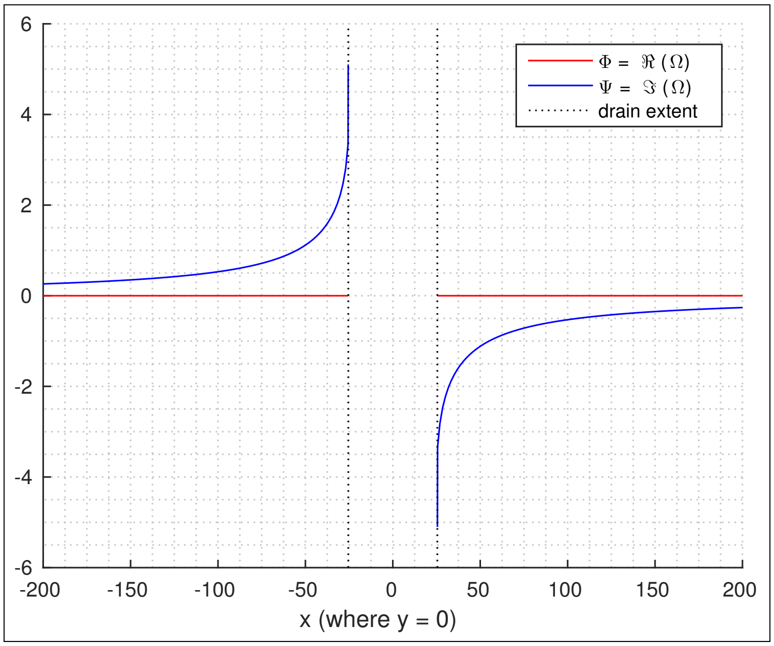

Figure 4: Boundary check in z-plane

- This verification process is coded in Matlab® as follows:

```
% --- check boundary conditions ------------------------------------------

tol = 1e-8;                                       % error tolerance (absolute)

% --- point 3 ---
point_3_zeta = -1;
point_3_z = z_of_zeta( point_3_zeta, k, N ,abs_A,abs_B );
point_3_PSI = imag(Omega_of_zeta(point_3_zeta, k, N,abs_A,abs_B));
assert (abs(point_3_PSI - (-N*point_3_z)) < tol,'point 3 bc not met');

% --- point 5 ---
point_5_zeta = 1;
point_5_z = z_of_zeta( point_5_zeta, k, N ,abs_A,abs_B );
point_5_PSI = imag(Omega_of_zeta(point_5_zeta, k, N,abs_A,abs_B));
assert (abs(point_5_PSI - (-N*point_5_z)) < tol,'point 5 bc not met');

% --- point 4 ---
point_4_zeta = complex(0,0);
point_4_PSI = imag(Omega_of_zeta(point_4_zeta, k, N,abs_A,abs_B));
assert (point_4_PSI < tol, 'point 4 bc not met');

% --- left/right drain Phi -----------------------------------------------

bext = 10; %boundary extent
zeta_left_of_neg1 = linspace(-bext,-1,10*bext);
zeta_right_of_pos1 = linspace( 1, bext,10*bext);
Omega_left_of_neg1 = zeros(1,bext);
Omega_right_of_pos1 = zeros(1,bext);

for ii = 1:length(zeta_left_of_neg1)
Omega_left_of_neg1(ii) = Omega_of_zeta(zeta_left_of_neg1(ii), k, N,abs_A,abs_B);
Omega_right_of_pos1(ii) = Omega_of_zeta(zeta_right_of_pos1(ii), k, N,abs_A,abs_B);
assert(real(Omega_left_of_neg1(ii)) == 0,'left drain phi bc not met');
assert(real(Omega_right_of_pos1(ii)) == 0,'right drain phi bc not met');
end
```

2) Ω as a Direct Function of Parameters

Given that eqs. 2 and 3 for $|A|$ and $|B|$ are expressed in terms of the parameters L, k and N, they can be substituted into eq. 1 to develop a total function for $\Omega = f\left(\zeta, L, k, N\right)$ as follows:

$$\Omega = f\left(\zeta, |A|, |B|\right) = -i|A|\frac{N}{N-k}\zeta + i|B|\frac{k}{N-k}\sqrt{(\zeta-1)(\zeta+1)}$$

$$|A| = f\left(L, k, N\right) = \frac{L}{2}k\frac{\sqrt{k-N}}{\sqrt{k+N}}$$

$$|B| = f\left(L, k, N\right) = \frac{L}{2}N\frac{\sqrt{k-N}}{\sqrt{k+N}}$$

$$\rightarrow \boxed{\Omega = f(\zeta, L, k, N) = -i\left(\frac{L}{2}k\frac{\sqrt{k-N}}{\sqrt{k+N}}\right)\frac{N}{N-k}\zeta + i\left(\frac{L}{2}N\frac{\sqrt{k-N}}{\sqrt{k+N}}\right)\frac{k}{N-k}\sqrt{(\zeta-1)(\zeta+1)}}$$

3) Flownets

3.1) ζ - plane

Figure 5 presents the flownet of the analysis plane mapped as $\Omega = f(\zeta)$. It was developed by separately contouring the real and imaginary portions of eq. 1 across a grid of ζ shown by the figure axes. The parameters used for the flow shown are:

- $L = 50m$
- $k = 1\frac{m}{\text{day}}$
- $N = 0.2\frac{m}{day}$

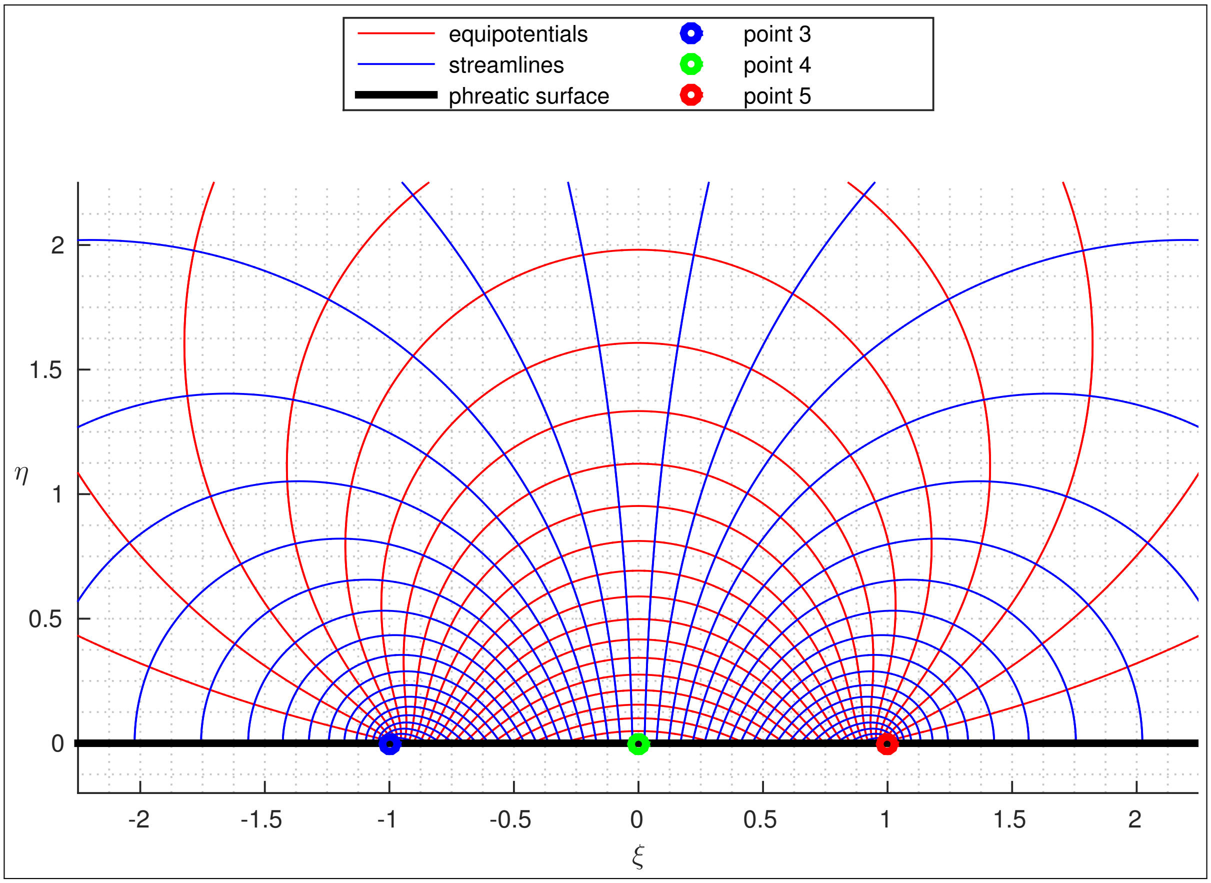

Figure 5: ζ-plane flownet

3.2) z - plane

Figure 6 presents the same complex discharge information as in figure 5 but mapped onto the physical z-plane using eq. 5. This mapping matches with that shown in introductory figure 2. It should be noted that in the ζ-plane, all positive and negative values of η have negative and positive x values, respectively, in the z-plane. This is also the case for all $\pm\xi$ having corresponding values of $\mp y$.

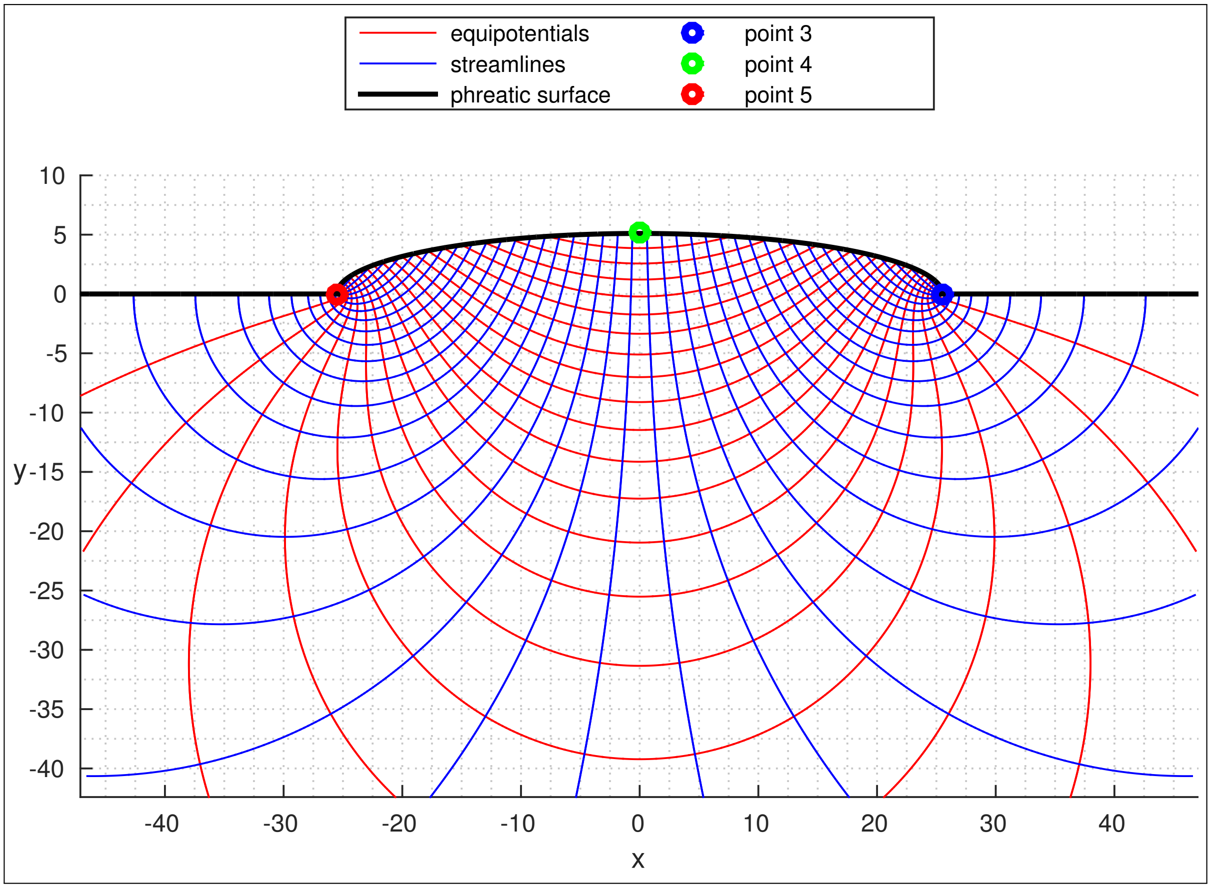

Figure 6: z-plane flownet

4) Discussion

This method allows for a flownet to be developed along the vertical cross section of the system of interest when the phreatic surface behaves as a free boundary. While this lack of constraint poses a difficulty, it allows the hodograph method to be used to develop a new complex potential equation as a function of the anti-conformally mapped ζ.

As shown in figures 7 through 10, a greater ${}^{N}/_{k}$ ratio causes the surface to "bulge" upwards, as expected.

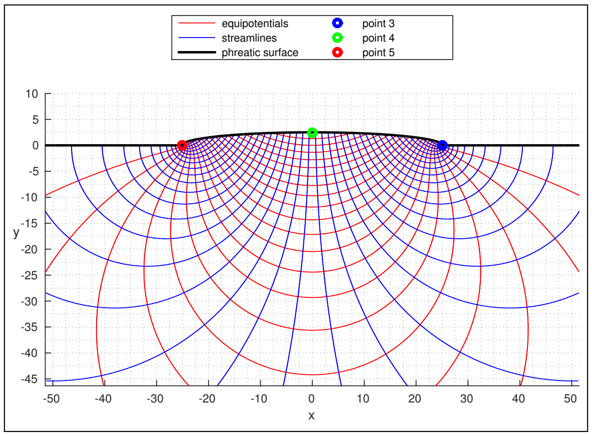

Figure 7: z-plane flownet for ${}^{N}/_{k} = 0.1$

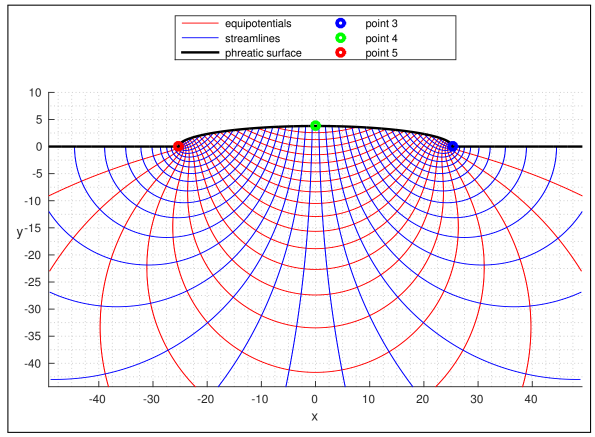

Figure 9: z-plane flownet for ${}^{N}/_{k} = 0.15$

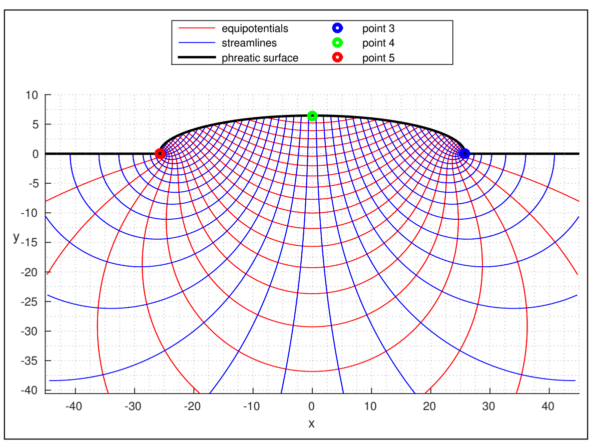

Figure 8: z-plane flownet for ${}^{N}/_{k} = 0.25$

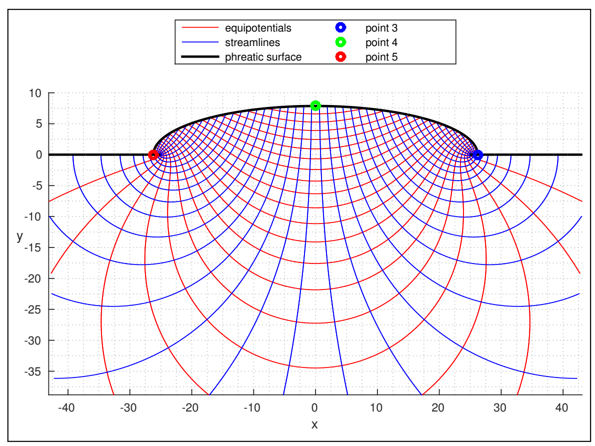

Figure 10: z-plane flownet for ${}^{N}/_{k} = 0.3$

As shown in table 1, the infiltration-hydraulic conductivity ratio ${}^{N}/_{k}$ causes the phreatic surface width ($2b$) to increase as well. As mentioned before, it should be noted that the limit of this model is all cases for which the infiltration rate exceeds hydraulic conductivity.

$$\frac{N}{k} < 1$$

N/k	$\mathbf{z_5}$	$\mathbf{z_3}$	**2b**
0.10	-25.1259 + 0i	25.1259 + 0i	50.2518
0.15	-25.2861 + 0i	25.2861 + 0i	50.5722
0.20	-25.5155 + 0i	25.5155 + 0i	51.0310
0.25	-25.8199 + 0i	25.8199 + 0i	51.6398
0.30	-26.2071 + 0i	26.2071 + 0i	52.4142

Table 1: Affect of N on point $4-5$ spacing

A) Appendix

A.1) Master Script

```
clc
close all
clear all

% --- given parameters ---------------------------------------------------

L = 50;                                          % length between drains [m]
k = 1;                                      % hydraulic conductivity [m/day]
N = 0.2;                                             % infiltration [m/day]

% --- preliminary calcs -------------------------------------------------

abs_A= 0.5*L*k*sqrt(k-N)/sqrt(k+N);
abs_B= 0.5*L*N*sqrt(k-N)/sqrt(k+N);

% --- check boundary conditions -----------------------------------------

tol = 1e-8;                                      % error tolerance (absolute)

% --- point 3 ---
point_3_zeta = -1;
point_3_z = z_of_zeta( point_3_zeta, k, N ,abs_A,abs_B );
point_3_PSI = imag(Omega_of_zeta(point_3_zeta, k, N,abs_A,abs_B));
assert (abs(point_3_PSI - (-N*point_3_z)) < tol,'point 3 bc not met');

% --- point 5 ---
point_5_zeta = 1;
point_5_z = z_of_zeta( point_5_zeta, k, N ,abs_A,abs_B );
point_5_PSI = imag(Omega_of_zeta(point_5_zeta, k, N,abs_A,abs_B));
assert (abs(point_5_PSI - (-N*point_5_z)) < tol,'point 5 bc not met');

% --- point 4 ---
point_4_zeta = complex(0,0);
point_4_PSI = imag(Omega_of_zeta(point_4_zeta, k, N,abs_A,abs_B));
assert (point_4_PSI < tol, 'point 4 bc not met');

% --- pre/post drain Phi ------------------------------------------------

bext = 10; %boundary extent
zeta_left_of_neg1 = linspace(-bext,-1,10*bext);
zeta_right_of_pos1 = linspace(1, bext,10*bext);
Omega_left_of_neg1 = zeros(1,bext);
Omega_right_of_pos1 = zeros(1,bext);

for ii = 1:length(zeta_left_of_neg1)
Omega_left_of_neg1(ii) = Omega_of_zeta(zeta_left_of_neg1(ii), ...
                                                   k, N,abs_A,abs_B);
Omega_right_of_pos1(ii) = Omega_of_zeta(zeta_right_of_pos1(ii), ...
                                                   k, N,abs_A,abs_B);
assert(real(Omega_left_of_neg1(ii)) == 0,'left drain phi bc not met');
assert(real(Omega_right_of_pos1(ii)) == 0,'right drain phi bc not met');
end

  % --- zeta plane plot ------------------------------------------------
figure; hold on; grid minor;
h1 = plot(zeta_left_of_neg1,real(Omega_left_of_neg1),'-r');
     plot(zeta_right_of_pos1,real(Omega_right_of_pos1),'-r');
h2 = plot(zeta_left_of_neg1,imag(Omega_left_of_neg1),'-b');
     plot(zeta_right_of_pos1,imag(Omega_right_of_pos1),'-b');
h3 = plot([-1 -1],[-20 20],':k');
plot([ 1  1],[-20 20],':k')
xlabel('\xi (where \eta = 0)');
axis([-bext, bext, -6, 6])
legend([h1 h2 h3],{'\Phi = \Re (\Omega)','\Psi = \Im (\Omega)',...
     'drain extent'},'location','northeast'); hold off;
print('103','-depsc2','-r300');

   % --- z plane plot ---------------------------------------------------
x_left_of_neg1  = zeros(1,length(zeta_left_of_neg1));
x_right_of_pos1 = zeros(1,length(zeta_right_of_pos1));

for ii = 1:length(x_left_of_neg1)
x_left_of_neg1(ii) = z_of_zeta(zeta_left_of_neg1 (ii) , k, N ,abs_A,abs_B );
x_right_of_pos1(ii) = z_of_zeta(zeta_right_of_pos1(ii),k,N,abs_A,abs_B);
end

figure; hold on; grid minor;
h1 = plot(x_left_of_neg1, real(Omega_left_of_neg1),'r');
plot(x_right_of_pos1,real(Omega_right_of_pos1),'r');
h2 = plot(x_left_of_neg1, imag(Omega_left_of_neg1),'b');
plot(x_right_of_pos1,imag(Omega_right_of_pos1),'b');
h3 = plot([z_of_zeta(1, k, N ,abs_A,abs_B ) ...
            z_of_zeta(1, k, N,abs_A,abs_B )],[-20 20],':k');
     plot([z_of_zeta(-1, k, N ,abs_A,abs_B ) ...
            z_of_zeta(-1, k, N,abs_A,abs_B )],[-20 20],':k');
xlabel('x (where y = 0)');
axis([-200 200 -6 6])
legend([h1 h2 h3],{'\Phi = \Re (\Omega)','\Psi = \Im (\Omega)',...
     'drain extent'},'location','northeast'); hold off;
print('104','-depsc2','-r300');

% --- plot flownet ------------------------------------------------------

wind = 2.25;
Nxy = 300;
nint = 40;

ContourMe_flow_net(-wind, wind, Nxy, 0 , wind, Nxy,...
    @(zeta)Omega_of_zeta( zeta, k, N,abs_A,abs_B),nint,k,N,abs_A,abs_B);
```

A.2) Function Omega $= \mathbf{f}(\zeta, \mathbf{k}, \mathbf{N}, |\mathbf{A}|, |\mathbf{B}|)$

```
function [ Omega_out] = Omega_of_zeta( zeta, k, N,abs_A,abs_B)

Omega_out = -1i*abs_A*N/(N-k)*zeta+...
                                 1i*abs_B*k/(N-k)*sqrt(zeta-1)*sqrt(zeta+1);


                            end
```

A.3) Function z $= \mathbf{f}(\zeta, \mathbf{k}, \mathbf{N}, |\mathbf{A}|, |\mathbf{B}|)$

```
function [ z_out ] = z_of_zeta( zeta, k, N ,abs_A,abs_B )

z_out = -abs_A*zeta/(k-N) + abs_B/(k-N)*sqrt(zeta-1)*sqrt(zeta+1);

end
```

A.4) Contouring Routine

```
function [Grid,zz] = ContourMe_flow_net(xfrom, xto, Nx, yfrom, yto, Ny,...
                                              func,nint,k,N,abs_A,abs_B)
if Nx ~= Ny
disp('z from zeta transformation assumes same quanity of Nx and Ny');
end

Grid = zeros(Ny,Nx);
X_zeta = linspace(xfrom, xto, Nx);
Y_zeta = linspace(yfrom, yto, Ny);
zeta = zeros(Nx,Ny);

for row = 1:Ny
    for col = 1:Nx
        Grid(row,col) = func(complex( X_zeta(col), Y_zeta(row)));
        zeta(row,col) = complex(X_zeta(col),Y_zeta(row));
    end
end

Bmax=max(imag(Grid));
Bmin=min(imag(Grid));
Cmax=max(Bmax);
Cmin=min(Bmin);
D=Cmax-Cmin;
del=D/nint;
Bmax=max(real(Grid));
Bmin=min(real(Grid));
Cmax=max(Bmax);
Cmin=min(Bmin);
D=Cmax-Cmin;
nintr=round(D/del);

% --- zeta plane flow net ------------------------------------------------
figure; hold on; axis square; axis equal; grid minor
contour(X_zeta, Y_zeta,real(Grid),nintr,'r');
contour(X_zeta, Y_zeta,imag(Grid),nint,'b');
xlabel('\xi'); ylabel('\eta','rot',0);

h3_zeta = plot([xfrom xto],[0 0],'-k','linewidth',3);
hp5_zeta =   plot(1,0,'ro','linewidth',3);
hp3_zeta =  plot(-1,0,'bo','linewidth',3);
hp4_zeta =   plot(0,0,'go','linewidth',3);

axis([xfrom, xto, -0.2, yto]);

h1_zeta = plot(NaN,NaN,'-r');
h2_zeta = plot(NaN,NaN,'-b');

gridLegend([h1_zeta h2_zeta h3_zeta hp3_zeta hp4_zeta hp5_zeta],2,...
{'equipotentials','streamlines','phreatic surface',...
                                           'point 3','point 4','point 5'});

hold off; print('101','-depsc2','-r300');

% --- z plane flownet ----------------------------------------------------
zz = zeros(Nx,Ny);
for row = 1:Ny
    for col = 1:Nx
        zz(row,col) = z_of_zeta(zeta(row,col), k, N, abs_A,abs_B  );
    end
end

figure; hold on; axis square; axis equal; grid minor
contour(real(zz), imag(zz),real(Grid),nintr,'r');
contour(real(zz), imag(zz),imag(Grid),nint,'b');

% --- phreatic surface ---------------------------------------------------
zeta_phreatic = linspace(xfrom,xto,100000);
zz_phreatic = zeros(1,length(zeta_phreatic));

for kk = 1:length(zz_phreatic)
zz_phreatic(kk) = z_of_zeta( zeta_phreatic(kk), k, N, abs_A,abs_B  ) ;
end
h3_z = plot(real(zz_phreatic),imag(zz_phreatic),'-k','linewidth',2);

% --- points of interest -------------------------------------------------
point_3_zeta = -1;
point_5_zeta =  1;
point_3_z = z_of_zeta( point_3_zeta, k, N ,abs_A,abs_B );
point_5_z = z_of_zeta( point_5_zeta, k, N ,abs_A,abs_B );
point_4_z = z_of_zeta( complex(0,0), k, N ,abs_A,abs_B );

hp5_z =  plot(real(point_5_z),imag(point_5_z),'ro','linewidth',3);
hp3_z =  plot(real(point_3_z),imag(point_3_z),'bo','linewidth',3);
hp4_z =  plot(real(point_4_z),imag(point_4_z),'go','linewidth',3);

% --- axis and legend ----------------------------------------------------
axis([z_of_zeta(xto  , k, N, abs_A,abs_B  )...
      z_of_zeta(xfrom, k, N, abs_A,abs_B   )...
      0.9*z_of_zeta(yto  , k, N, abs_A,abs_B   ) 10]);

h1_z = plot(NaN,NaN,'-r');
h2_z = plot(NaN,NaN,'-b');
xlabel('x'); ylabel('y','rot',0);

gridLegend([h1_z h2_z h3_z hp3_z hp4_z hp5_z],2,...
    {'equipotentials','streamlines','phreatic surface',...
    'point 3','point 4','point 5'});

hold off; print('102','-depsc2','-r300');
end
```